Subvarieties with Splitting Tangent Sequences

by

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Declaration

This dissertation is submitted to the University of Warwick in support of my application for the degree of Master of Science in Mathematics. It has been composed by myself and has not been submitted in any previous application for any degree. The work presented was carried out by the author.

Abstract

In this dissertation we prove the *Van de Ven theorem*: if X is a closed subvariety of $\mathbb{P}^n$. The short exact sequence

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{T}_{\mathbb{P}^n}|_X \longrightarrow \mathscr{N}_{X|\mathbb{P}^n} \longrightarrow 0$$

splits if and only if X is linear.

We will recall standard results about algebraic groups, following [Bor91], before tackling the *Matusmura-Oort* and *Borel-Remmert* theorems. Finally, we will prove the Van de Ven problem by following the work of Mustaţă and Popa.

0 Introduction

It is well known that for a closed embedding of smooth $\mathbb{C}$-varieties $X \hookrightarrow Y$ the *tangent sequence*

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{T}_Y|_X \longrightarrow \mathscr{N}_{X|Y} \longrightarrow 0 \qquad (\star)$$

is short exact. Here $\mathscr{T}_X$ and $\mathscr{N}_{X|Y}$ are the tangent and normal bundles of X in Y and $\mathscr{T}_Y|_X$ is the restricted tangent bundle of Y to X. A natural next step is to ask when $(\star)$ *splits,* when is $\mathscr{T}_Y|_X$ isomorphic to $\mathscr{T}_X \oplus \mathscr{N}_{X|Y}$? In the mid-twentieth century Van de Ven in [Ven58] proved the following result.

Theorem 0.0.1 (The Van de Ven Theorem for $\mathbb{P}^n$)**.** *For* $Y = \mathbb{P}^n$, *if* $(\star)$ *splits then* X *is a linear subvariety of* Y.

Morrow and Rossi gave a complex-geometric proof in [MR78] by explicitly constructing a holomorphic retraction from a neighbourhood of X in $\mathbb{P}^n$ to X, together with the following corollary of Bézout's theorem:

Corollary 0.0.2. *If* L *is a linear subspace of* $\mathbb{P}^n$ *which intersects X transversally in the non-empty linear space* $X \cap L$*. Then* X *must be a linear subspace of* $\mathbb{P}^n$.

Alternatively, by dualising $(\star)$, we obtain the *cotangent sequence*

$$0 \longrightarrow \mathscr{N}^{\vee}_{X|Y} \longrightarrow \Omega^1_Y|_X \longrightarrow \Omega^1_X \longrightarrow 0. \qquad (\star\star)$$

Of course, $(\star\star)$ splits if and only if $(\star)$ does. Simons showed in 1971 the following result

Theorem 0.0.3 ([Sim71])**.** *If* X *is linearly normal in* $\mathbb{P}^n$ *and the restricted cotangent bundle* $\Omega_{\mathbb{P}^n}|_X$ *splits, then* X *is a rational curve.*

One decade later Laksov presented a simplified version of Simons's proof and showed that his result implies the Van de Ven theorem. In 1996 Mustaţă and Popa found a similar proof by studying the conditions that the first infinitesimal neighbourhood $X^{(1)}$ of X, which is the proof we present.

Before tackling Van de Ven, we need a result by Borel and Remmert ([BR62]), the *Borel-Remmert* decomposition of a projective homogeneous variety X.

Theorem 0.0.4. *If* X *is a projective homogeneous variety then* X *decomposes into*

$$X \cong A \times G/P$$

where A *is an abelian variety and* G/P *is the quotient of an algebraic group* G *by a parabolic subgroup* P *yielding a rational, homogeneous and projective variety.*

The proof of Borel-Remmert has two prerequisits. Firstly, we must understand what it means for a variety to be of the form G/P. This is classic theory treated well in Borel's famous book [Bor91]. We cover the basics of algebraic groups and build up to our first key theorem

Theorem 0.0.5. *For a connected affine algebraic group* G, G/P *is projective if and only if* P *is a parabolic subgroup.*

The second preliminary to Borel-Remmert is the Matsumura-Oort theorem ([MO67]) published in 1967. Published is much greater generality than we need, we do not treat the proof itself but instead prove its application to our scenario, in particular we show

Theorem 0.0.6. *If* X *is a connected projective variety then* $\mathrm{Aut}(X)$ *is a connected algebraic group whose Lie algebra is* $\Gamma(X, \mathscr{T}_X)$.

In Chapter 3 we use Matsumura-Oort to show that a connected projective variety X is homogeneous if and only if

$$\underline{\mathrm{op}_X} : \mathscr{O}_X \otimes \Gamma(X, \mathscr{T}_X) \longrightarrow \mathscr{T}_X$$

is surjective. Here $\underline{\mathrm{op}_X}$ is a map of sheaves induced by the map of Lie algebras

$$\mathrm{op}_X : \mathfrak{g} \longrightarrow \Gamma(X, \mathscr{T}_X).$$

Combining this with Chevalley's structure theorem, we will arrive at a proof of the Borel-Remmert theorem, before finally tackling the Van de Ven theorem for $\mathbb{P}^n$.

1 Varieties and their Group Structures

The goal of this chapter is to describe how we can endow a complex algebraic variety with a group structure. We first introduce the notion of an *algebraic group*. When an algebraic group G acts on morphically on a variety X, we call X a G-*space.* Our goal is to classify precisely when quotients of these groups are projective or affine. Our main reference for this chapter is [Bor91].

1.1 G-spaces

Before defining a G-spaces, we need some basic properties of *algebraic groups.*

Definition 1.1.1. An *algebraic group* is a variety X equipped with a group structure such that the group operations

$$\begin{aligned} \mu : G \times G &\longrightarrow G & \qquad i : G &\longrightarrow G \\ (g, h) &\longmapsto gh & g &\longmapsto g^{-1} \end{aligned}$$

are morphisms of varieties. An algebraic group whose underlying variety is an affine variety is said to be an *affine algebraic group.*

Definition 1.1.2. Let G be an algebraic group. We say that G is *connected* if the corresponding variety is Zariski-connected. We will denote the connected component of e by G^0. Unless explicitly stated otherwise we may assume G is connected.

Ideally, we would like to work with abelian groups, however we cannot always do this, so instead we work with *soluble* groups, which can be constructed from abelian groups.

Definition 1.1.3. Let G be a group and let [G, G] denote the commutator subgroup. Define inductively $G^{(0)} = G, G^{(1)} = [G, G], \ldots, G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$. Then the descending series

$$G^{(0)} \triangleright G^{(1)} \triangleright \cdots$$

is called the *derived series* of G. If there exists an $m \in \mathbb{N}$ such that $G^{(m)} = \{e\}$, we say that G is *soluble.*

Definition 1.1.4. Let G be an algebraic group. A variety X is a G-*space* if there exists a group action

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g.x \end{aligned}$$

which is also a morphism of varieties. We say that G acts $\mathbb{C}$-*morphically,* and we call G a $\mathbb{C}$-*group.* Let X and Y be G-spaces, A *morphism of* G-*spaces* is a morphism of varieties $\varphi : X \to Y$ such that $\varphi(g.x) = g.\varphi(x)$ for all $g \in G, x \in X$, i.e. φ is a morphism of varieties that is *equivariant*. A G-space X is said to be *homogeneous* if G acts transitively on X. If G acts with only a dense orbit, we say X is *almost homogeneous.*

Example 1.1.5. The set of all $n \times n$ matrices over $\mathbb{C}$, $GL_n(\mathbb{C})$, can be given the structure of an affine variety. Let $A = (x_{ij}) \in GL_n(\mathbb{C})$. We identify $GL_n(\mathbb{C})$ with the closed set

$$\left\{(A, y) \in \mathbb{A}^{n^2}_{x_{ij}} \times \mathbb{A}_t \mid \det(A) = 1\right\} = \mathbb{V}(\det(x_{ij})t - 1) \subseteq \mathbb{A}^{n^2} \times \mathbb{A}.$$

Thus, $GL_n(\mathbb{C})$ is a linear algebraic group.

Definition 1.1.6. A Zariski-closed subgroup of $GL_n(\mathbb{C})$ is called a *linear algebraic group*.

Definition 1.1.7. Let V be a finite dimensional vector space over $\mathbb{C}$. A morphism $\varphi : G \to GL(V)$ which is also a group homomorphism is called a *rational representation.*

Definition 1.1.8. Let $G = \mathrm{Spec}(\mathbb{C}[G])$ be an affine algebraic group. Our goal with this definition is to formulate the elements and structure of G in terms of $\mathbb{C}[G]$. This is sometimes known as an associative *Hopf algebra.*

Firstly, for the identity $e \in G$, we associate the evaluation map at e.

$$\begin{aligned} e : \mathbb{C}[G] &\longrightarrow \mathbb{C} \\ e(f) &\longmapsto f(e). \end{aligned}$$

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For *comultiplication* $\Delta : G \to G \times G$, we have

$$\nabla : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \bigotimes_{\mathbb{C}} \mathbb{C}[G]$$

such that if $\nabla(f) = \sum g_i \otimes h_i$ then $f(xy) = \sum g_i(x)h_i(y)$ where ∇ denotes the *multiplication map.*

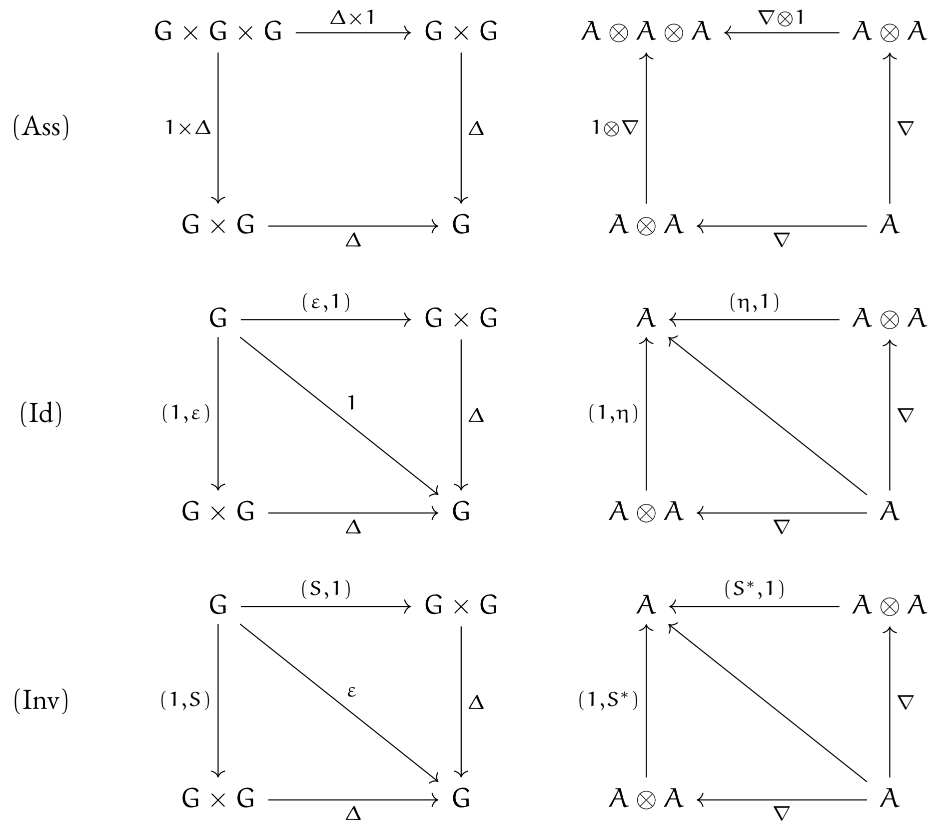
Finally, for $S : G \to G$ we have the *antipode* map

$$S^* : \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$$
$$(S^*f)(x) \longmapsto f\left(x^{-1}\right).$$

In order to formulate the group axioms, we first define the correspondence

$$\varepsilon : G \longrightarrow G \qquad \eta : A \longrightarrow A$$
$$x \longmapsto e \qquad \eta(x) \longmapsto f(e).$$

We call ε the *counit* and η the *unit* maps. The group axioms are thus expressed by the commutativity of the following diagrams:



Note that the right hand diagrams are guaranteed to commute if the left hand ones do (and vice-versa), as there is an equivalance of categories between the category of affine varieties and the category of finitely generated reduced $\mathbb{C}$-algebras.

Definition 1.1.9. Let G be an affine algebraic group which acts $\mathbb{C}$-morphically on an affine variety X via the map

$$\alpha : G \times X \longrightarrow X$$
$$(g, x) \longmapsto g.x.$$

Such an action makes X a *Hopf algebra comodule.* We have an induced pullback on coordinate rings

$$\alpha^* : \mathbb{C}[X] \to \mathbb{C}[G] \bigotimes_{\mathbb{C}} \mathbb{C}[X].$$

For $g \in G$ we denote λ_g for the pullback of $x \mapsto g^{-1}x$ and ρ_g for the pullback of $x \mapsto xg$ we thus have linear automorphisms of $\mathbb{C}[X]$ given by

$$f \longmapsto \lambda_g f \qquad (\lambda_g f)(x) = f(g^{-1}x)$$
$$f \longmapsto \rho_g f \qquad (\rho_g f)(x) = f(xg).$$

We call such automorphisms, respectively, *left translations* or *right translations* of functions by g.

Borel uses these translations to prove the following important theorem:

Theorem 1.1.10. *Let* G *b an affine* $\mathbb{C}$*-group. Then* G *is isomorphic to a closed subgroup of some* GL_n.

We now start building to our first important result; showing that for a linear algebraic group G and closed subgroup H the quotient G/H is quasi-projective. We begin with some easy results about oribts and stabilisers.

Lemma 1.1.11. *Let* X *be a* G*-space. Then, for each* $x \in X$, *the stabiliser subgroup*

$$G_x \coloneqq \{g \in G \mid g.x = x\}$$

is closed.

Proof. Let $\varphi : G \to X$ be the orbit map sending g to $g.x$. By the definition of G-spaces, we know that φ is a closed map so $\varphi^{-1}(x) = G_x$ is closed. □

Lemma 1.1.12. *Let* X *be a* G*-space. Then, for all* $x \in X$ *its orbit,* $G.x$ *is open in its closure.*

Proof. As $G.x$ is a morphism of varieties, it is open in its closure. □

Lemma 1.1.13 (Chevalley)**.** *Let* G *be a linear algebraic group and* H *a closed subgroup. There is a rational representation* $\alpha : G \to GL_n(V)$ *and a line* $L \subseteq V$ *such that*

$$H = \{g \in G \mid \alpha(g)L = L\}$$

i.e. H *is the stabiliser subgroup of* L.

Theorem 1.1.14. *Let* G *be a linear algebraic group over* $\mathbb{C}$ *and let* $H \leqslant G$ *be a closed subgroup. Then* G/H *is a quasi-projective variety.*

Proof. By Lemma 1.1.13, we can choose $\bar{v}$ to be the point in $\mathbb{P}^n$ corresponding to the line $\langle v \rangle$ stabilised by H. Let $X = G.\bar{v} \subseteq \mathbb{P}^n_k$. By construction, X is a homogeneous G-space. We have the induced surjective map

$$\varphi : G \longrightarrow X$$
$$g \longmapsto g.\bar{v},$$

whose fibres are cosets of H. This further induces the bijective map $\overline{\varphi} : G/H \to X$, which gives G/H a variety structure. We know that $\overline{G.\bar{v}} \subseteq \mathbb{P}^n_k$ is closed and therefore projective. Furthermore, by Lemma 1.1.12, $G.\bar{v}$ is open in $\overline{G.\bar{v}}$, therefore $G.\bar{v}$ is quasi-projective. □

1.2 PARABOLIC AND BOREL SUBGROUPS

Throughout this section, G is an algebraic group. We have two goals in this section. Firstly, we wish to classify when quotients of G are projective or affine. Secondly, we will classify parabolic subgroups of a simple algebraic group G, up to conjugacy.

Definition 1.2.1. A *complete variety* is a variety X such that for any variety Y, the projection $\pi : X \times Y \to Y$ is a closed map.

Theorem 1.2.2. *If* X *is projective then* X *is complete.*

Proof. We start by reducing what we need to prove. Firstly, (from the definition of a complete variety) if X is complete and $X' \subseteq X$ is a closed subvariety, X' is complete. Hence, we may let $X = \mathbb{P}^n$. Furthermore, as closed is a local condition it suffices to prove the theorem for each affine chart of Y. However, if $Y' \subseteq Y$ is an affine subvariety $X \times Y'$ is closed in $X \times Y$, so it suffices to prove the theorem for $Y = \mathbb{A}^m$.

Let π be the projection map $\pi : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$. We need to show that if Z is closed in $\mathbb{P}^n \times \mathbb{A}^m$ then $\pi(Z)$ is closed in $\mathbb{A}^m$. It is clear that for $Z = \varnothing$ that $\pi(Z)$ is closed, so we may assume that Z is non-empty. Now we can write Z as a finite union of closed Z_i's in $\mathbb{P}^n \times \mathbb{A}^m$ and show that each $\pi(Z_i)$ is closed. Hence, we can assume Z is irreducible.

Let $A = k[x_1, \dots, x_m]$ and $S = A[t_0, \dots, t_n]$. By the definition of the Zariski topology, the closed sets in $\mathbb{P}^n \times \mathbb{A}^m$ are of the form $\mathbb{V}(I)$ where I is a homogeneous ideal in S. So $Z = \mathbb{V}(I)$ is irreducible if and only if $\sqrt{I}$ is prime. W.L.O.G we can take I prime. Note that the restriction of π is dense in its image, so we can assume π is dominant.

It remains to show the following: if $\underline{y} = (y_1, ..., y_m) \in \mathbb{A}^m$ then there is an $\underline{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n$ such that $(\underline{x}, \underline{y}) \in \mathbb{V}(I)$. Let $\mathfrak{m}_{\underline{y}}$ be the maximal ideal in A corresponding to the point $\underline{y}$. Then $J = \mathfrak{m}_{\underline{y}} S + I$ is a proper homogeneous ideal in S. The goal is to show that $\mathbb{V}(J) \neq \varnothing$.

We argue by contradiction. Assume $\mathbb{V}(J) = \varnothing$. Then, there exists a d such that the set $S_d \subset S$ of homogeneous polynomials in degree d lies in J. Let $N = \frac{S_d}{S_d \cap I}$, which is a finite A-module. Let $(n_1, ..., n_r)$ be its generators. It follows from our assumptions that $N = \mathfrak{m}_{\underline{y}} N$. By Nakayama's lemma, $N = 0$, which implies that $\mathbb{V}(J) \neq \varnothing$ contradicting our assumption. □

Definition 1.2.3. A closed subgroup P of G is *parabolic* if G/P is a complete variety.

Theorem 1.2.4. *Let* G *be a linear algebraic group. A variety* G/P *is projective if and only if* P *is parabolic.*

Proof. The forwards direction follows from the definition of a parabolic subgroup and Theorem 1.2.2. For the other direction, by Theorem 1.1.14 G/P is always quasi-projective. As G/P is also complete, it is projective. □

Now we know G/P is projective, we can work to more the more general case G/B where B is *Borel*.

Definition 1.2.5. A subgroup B of G, which is maximal among the connected, soluble subgroups is called a *Borel subgroup.*

Theorem 1.2.6 (Borel fixed point theorem)**.** *Let* G *be a connected, soluble, linear algebraic group acting on a non-empty complete variety* X. *Then* X *has a fixed point under the action of* G.

Theorem 1.2.7. *Let* B *be a Borel subgroup of a linear algebraic group. Then all Borel subgroups are conjugate to* B, *and* G/B *is a projective variety.*

Corollary 1.2.8. *A closed subgroup* P *of* G *is parabolic if and only if* P *contains a Borel subgroup.*

Proof. If P contains a Borel subgroup B, then we have a surjective morphism $G/B \to G/P$ from a projective variety (Theorem 1.2.7), so G/P is projective. For the other direction, by Theorem 1.2.6, B has a fixed point in G/P so some conjugate of B lies in P. □

From the previous corollary, we see that Borel subgroups are themselves parabolic and are in fact the minimal parabolic subgroups.

Definition 1.2.9. A linear algebraic group over $\mathbb{C}$ is said to be *reductive* if it has a representation that has a finite kernel and is a direct sum of irreducible representations.

Theorem 1.2.10 (Matsushima's Criterion)**.** *Let* G *be a reductive linear algebraic group and let* H *be a closed subgroup, then* G/H *is affine if and only if* H *is reductive.*

Proof. See [Bia63] □

1.3 INTERLUDE ON LIE ALGEBRAS

We briefly discuss the associated Lie algebras of algebraic groups and their Borel/parabolic subgroups to see how the classification of semisimple Lie algebras can be used to classify parabolic subgroups.

Recall that for a semisimple Lie algebra $\mathfrak{g}$ a choice of Cartan subalgebra $\mathfrak{h}$ determines the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where R is the root system.

Definition 1.3.1. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. For each ordering of the root system $R = R^+ \cup R^-$ we associate the subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

called the *Borel subalgebra*.

If G is algebraic group corresponding to the Lie algebra g and P a parabolic subgroup of G, we can describe the Lie algebra as follows: Theorem 1.2.7 tells us that a Borel subgroup B must live inside it. Hence, if T is a subset of the root system R, containing R^+ we have a subspace of $\mathfrak{g}$ containing $\mathfrak{b}$

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}_\alpha.$$

For $\mathfrak{p}$ to be a subalgebra, we require T to be closed under addition. Moreover, we can see that T must be generated by R^+ and the negatives of a set $\Sigma \subseteq R$. Hence, for each Σ we write $T(\Sigma)$ for the set of all roots which can be written as $-s + r$ for $s \in \Sigma, r \in R^+$. We are now ready for our final definition in this section.

Definition 1.3.2. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and Σ an additively closed subset of the root system R. We define the *parabolic subalgebra* with respect to Σ as

$$\mathfrak{p}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in T(\Sigma)} \mathfrak{g}_{\alpha}.$$

We of course have corresponding parabolic subgroups $P(\Sigma)$ of the corresponding algebraic group.

Thus, we conclude this section by noting that there is a one-to-one correspondence between the subsets of the set of simple roots (or equivalently with subsets of nodes of the Dynkin diagram) and the parabolic subgroups of a simple algebraic group G.

1.4 Application: The Group Representation to Line Bundle Correspondence

In the first part of this section we recall the definitions of *schemes* and $\mathcal{O}_X$*-modules* and finally, the definition of a *line bundle.* In the second part, we describe how, for homogeneous spaces $X = G/P$, there is a one-to-one correspondence between (ample) line bundles $\mathscr{L}$ on X and irreducible representations V of G. We will roughly follow [FH91].

1.4.1 Line Bundles

Definition 1.4.1. Let X be a topological space, and $\mathcal{O}_X$ a sheaf of rings. We call $(X, \mathcal{O}_X)$ a *ringed space*. $(X, \mathcal{O}_X)$ is a *locally ringed space* if for each $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

Definition 1.4.2. An *affine scheme* is a locally ringed space which is isomorphic to the spectrum of some ring. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that for each $p \in X$ there exists a neighbourhood $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. We will often call $\mathcal{O}_X$ the *structure sheaf* of X. A morphism of schemes is a morhpism of locally ringed spaces.

Definition 1.4.3. Let $(X, \mathcal{O}_X)$ be a ringed space. We define an $\mathcal{O}_X$*-module* as a sheaf of abelian groups $\mathscr{F}$ such that, for each $U \subseteq X$, $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module and $\mathscr{F}$ is compatible with the restriction maps of $\mathscr{O}_X$: if $U \subseteq V \subseteq X$ then the diagram

$$\begin{array}{ccc} \mathscr{O}_X(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \\ \Big\downarrow{\scriptstyle (\mathrm{res}_{V,U}, \mathrm{res}_{V,U})} & & \Big\downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathscr{O}_X(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \end{array}$$

commutes.

Definition 1.4.4. Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. For an $\mathscr{O}_Y$-module $\mathscr{G}$ define the *pullback* of $\mathscr{G}$ as

$$f^*\mathscr{G} \coloneqq f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

Definition 1.4.5. A *free sheaf* on a ringed space $(X, \mathscr{O}_X)$ is an $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{F} \cong \mathscr{O}_X^{\oplus I}$ for some indexing set I. The cardinality of I is the *rank* of $\mathscr{F}$.

A *locally free sheaf* is an $\mathscr{O}_X$-module which locally isomorphic to a free sheaf.

The most important type of sheaf that we need is the *invertible sheaf*, also known as (albeit slightly informally) a *line bundle*.

Definition 1.4.6. A *line bundle* or *invertible sheaf* is a locally free sheaf of rank 1.

1.4.2 The Correspondence

Let G be a simple homogeneous algebraic group. Let $V = \Gamma_\lambda$ be an irreducible representation of G with height weight λ. Consider the action of G on $\mathbb{P}V$ and let $p \in \mathbb{P}V$ be the point corresponding to the eigenspace with eigenvalue λ.

Proposition 1.4.7. The orbit $G.p$ is the (unique) closed orbit of the action of G on $\mathbb{P}V$.

Proof. The point p is fixed by a Borel subgroup B such that $\mathrm{Stab}(p) = P_\lambda$ for P_λ parabolic. So the orbit G/P_λ is compact and thus closed.

On the other hand, by Theorem 1.2.6, any closed orbit of G has a fixed point for the action of B, but we defined p as the unique fixed point of B acting on $\mathbb{P}V$. □

We have shown that for any irreducible representation V of G there is a unique, closed orbit $X = G/P$ for the action of G on $\mathbb{P}V$. Moreover, from a representation V we obtain a homogeneous projective variety X and, by restricting the tautological line bundle on $\mathbb{P}V$, we obtain a line bundle $\mathscr{L} = \mathscr{O}_{\mathbb{P}V}|_X$ on X which is invariant under the action of G.

More generally, via the projection $\pi : G/B \to G/P$ we can pull back each line bundle $\mathscr{L}$ on X to a line bundle $\pi^*\mathscr{L}$ on G/B.

1.5 Examples

There is a natural way to turn $SL_n(\mathbb{C})$ into a linear algebraic group.

Example 1.5.1. The group $SL_n(\mathbb{C})$ of determinant 1 matrices form an affine variety in the following way: Let Δ denote the determinant

$$\Delta(x_{ij}) = \det \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}.$$

It is clear that this is a polynomial in the n^2 variables x_{ij}. The affine subvariety

$$\mathbb{V}(\Delta - 1) \subset \mathbb{A}^{n^2}$$

gives $SL_n(\mathbb{C})$ a variety structure.

Example 1.5.2. The group of upper triangular matrices Borel in $SL_n(\mathbb{C})$ is Borel. For this choice of Borel subgroup, the maximal parabolic subgroups are of the form

$$P_d = \left\{ \begin{pmatrix} * & * \\ 0_{(n-d)\times d} & * \end{pmatrix} \in SL_n(\mathbb{C}) \right\}, \quad 1 \leqslant d \leqslant n-1.$$

More generally, let $\underline{d} = (d_1, ..., d_r)$ where $1 \leqslant d_1 \leqslant \cdots \leqslant d_r \leqslant n-1$ for some $1 \leqslant r \leqslant n-1$. Then, a general parabolic subgroup of $SL_n(\mathbb{C})$ is of the form

$$P_{\underline{d}} = \bigcap_{i=1}^{r} P_{d_i}.$$

1.5.1 Grassmannians and Flag Varieties

Definition 1.5.3. Let V be a finite dimensional vector space of dimension n. A *flag* is a sequence of nested subspaces of V,

$$V_1 \subset \cdots \subset V_r \subset V.$$

The *signature* of a flag is the tuple of dimensions $\sigma = (\dim(V_1), \ldots, \dim(V_r))$. If $\sigma = (1, \ldots, n)$ we have a *full flag*.

Definition 1.5.4. Let V be a finite dimensional vector space of dimension n. The set of flags of V, with signature $(d_1, \ldots, d_r)$, denoted $\mathcal{F}(d_1, \ldots, d_r, n)$ is called the *flag variety* of V. We let $\mathcal{F}(n)$ denote the set of all flags of V. Finally, if each V_i in

$$V_1 \subset \cdots \subset V_r$$

is the span of the first r basis vectors in the standard basis, $\mathcal{F}(1, \dots, r, n)$ is called the *standard flag*.

We note that $GL_n(\mathbb{C})$ acts transitively on $\mathcal{F}_n$, i.e. if $g \in GL_n(\mathbb{C})$ and $x = V_1 \subset \dots \subset V_r \in \mathcal{F}_n$ then

$$g.x = gV_1 \subset \cdots gV_r.$$

Furthermore, let g be in the stabiliser of the standard flag, that is

$$gV_i \subset V_i \quad \text{for all } 1 \leqslant i \leqslant n.$$

More precisely, the i'th basis vector must land in the span of the $e_1, \dots, e_i$. We can see that the standard flag is stabilised by elements of the form

$$\begin{pmatrix} x_{11} & x_{11} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{pmatrix}.$$

In other words, the standard flag is stabilised by the group of upper triangular matrices which we will denote $B_n(\mathbb{C})$. Using an induction argument, it is clear that $B_n(\mathbb{C})$ is soluble.

Definition 1.5.5. Let V be an n-dimensional vector space. The *Grassmannian* $G(k, V) = G(k, n)$, is the set of k-dimensional subspaces of V. We can embed $G(k, n)$ as a projective variety into projective space via the *Plücker embedding:*

$$\begin{aligned} \rho : G(k, n) &\longrightarrow \mathbb{P}\Big(\bigwedge^k V\Big) \cong \mathbb{P}^{\binom{n}{k}-1} \\ W = \mathrm{Span}(\{v_1, \dots, v_k\}) &\longmapsto v_1 \wedge \cdots \wedge v_k. \end{aligned}$$

Suppose we have a representation of $SL_n(\mathbb{C})$ in the form $W = \bigwedge^k V$, where V is the standard representation, Proposition 1.4.7 tells us that the vectors $\{v_1 \wedge \cdots \wedge v_k\}$ form a closed orbit in $\mathbb{P}W$. Hence, we can embed $SL_n(\mathbb{C})/P$ into the Grassmannian.

Recall that $\mathcal{F}(d_1, \dots, d_r, n)$ is the flag of signature $(d_1, \dots, d_r)$ in a vector space V of dimension n. We can embed a flag variety into projective space as follows: The elements of $\mathcal{F}(d_1, \dots, d_r, n)$ are of the form

$$V_{d_1} \subset \cdots \subset V_{d_r}$$

where V_{d_i} is a vector space of dimension d_i. There is an embedding

$$\begin{aligned} \varphi : \mathcal{F}(d_1, \dots, d_r, n) &\longrightarrow \prod_{i=1}^{r} G(d_i, n) \\ V_{d_1} \subset \cdots \subset V_{d_r} &\longmapsto (V_{d_1}, \dots, V_{d_r}) \end{aligned}$$

into a product of Grassmannians. We can then use a product of Plücker embeddings

$$\rho_{(d_1,\ldots,d_r)} : \prod_{i=1}^{r} G(d_i, n) \hookrightarrow \prod_{i=1}^{r} \mathbb{P}\Big(\bigwedge^{d_i} V\Big) \cong \prod_{i=1}^{r} \mathbb{P}^{\binom{n}{d_i}-1}$$
$$W_1 \times \cdots \times W_r \longmapsto \wedge^{d_1} W_1 \times \cdots \times \wedge^{d_r} W_r.$$

Putting this all together we have a map

$$\rho_{(d_1,\ldots,d_r)} \circ \varphi : \mathcal{F}(d_1, \ldots, d_r, n) \hookrightarrow \prod_{i=1}^{r} \mathbb{P}^{\binom{n}{d_i}-1}.$$

So any flag variety can be embedded into projective space. Moreover, every G/P, can be realised as a flag hence the following definition

Definition 1.5.6. A *generalised flag variety* is a smooth, projective homogeneous space G/P which is *full* when P is Borel.

The Matsumura-Oort Theorem 2

We wish to study the group of all automorphisms of a variety X, which we denote $\mathrm{Aut}(X)$. An important case is $X = \mathbb{P}^n$, for which it can be shown that

$$\mathrm{Aut}(X) = \mathrm{PGL}_{n+1}(\mathbb{C}).$$

Projective algebraic groups are automatically commutative if they are connected, hence we have the following definition:

Definition 2.0.1. An *abelian variety* is a connected projective algebraic group.

Abelian varieties arise in many areas, for example in the *Albanese* variety which we will treat in Chapter 3. Of course, the most famous type of abelian varieties are elliptic curves. Furthermore, for any abelian variety, we have the following property of its automorphism group.

Example 2.0.2. Let A be an abelian variety, G the group of automorphisms which preserve the abelian structure of A and $A(k)$ the group of translations of points in A. Then A is a group extension of G by $A(k)$. We obtain the sequence

$$1 \to A(k) \to \mathrm{Aut}(A) \to G \to 1$$

which is exact.

2.1 The Automorphism Group of $\mathbb{P}^n$

Showing that the automorphism group of $\mathbb{P}^n$ is $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a surprisingly difficult task. For this, we need the language of *schemes* and *line bundles.*

The most important line bundles for us is the following example. First, we borrow some notation from [Vak24] by setting $x_{j/i} \coloneqq \frac{x_j}{x_i}$ thus, we may denote the affine open set $U_i \subset \mathbb{P}^n$ as

$$U_i = \mathrm{Spec}\left(\frac{\mathbb{C}[x_{0/i}, \dots, x_{n/i}]}{(x_{i/i} - 1)}\right)$$

where $x_{i/i}$ is a “dummy variable.”

Example 2.1.1. Let $X = \mathbb{P}^n$. We define a line bundle $\mathscr{O}_X(m)$ (we will use the shorthand $\mathscr{O}(m)$) via transition functions as follows: The transition function U_i to U_j is multiplication by $x_{i/j}^m = x_{j/i}^{-m}$. For $m \geqslant 0$ We have the following diagram:

$$\mathbb{C}[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1) \underset{\times x_{i/j}^{-m}}{\overset{\times x_{j/i}^{-m}}{\rightleftarrows}} \mathbb{C}[x_{0/j}, \ldots, x_{n/j}]/(x_{j/j} - 1)$$

It is clear that $\mathscr{O}(m) = \mathscr{O}(1)^{\otimes m}$. If $m < 0$ we define $\mathscr{O}(m) = (\mathscr{O}(1)^{\otimes -m})^*$. Of particular importance to us are the line bundles $\mathscr{O}(1)$ and $\mathscr{O}(-1)$ which are often referred to as the *Serre twisting sheaf* and the *tautological line bundle* respectively.

We can quickly compute the global sections of $\mathscr{O}(m)$ as follows: let $m \geqslant 0$. Global sections are polynomials $f \in \frac{\mathbb{C}[x_{0/i}, \ldots, x_{n/i}]}{(x_{i/i} - 1)}$ and $g \in \frac{\mathbb{C}[x_{0/j}, \ldots, x_{n/j}]}{(x_{j/j} - 1)}$ such that

$$f\left(x_{0/i}, \ldots, \frac{1}{x_{i/j}}, \ldots, x_{n/i}\right) x_{i/j}^m = g(x_{0/j}, \ldots, x_{n/j})$$

i.e. we need $f\left(x_{0/i}, \ldots, \frac{1}{x_{i/j}}, \ldots, x_{n/i}\right) x_{i/j}^m$ to be a polynomial, which only happens when $f(x_{0/i}, \ldots, x_{n/i})$ is a polynomial of degree at most m. The "sticks and stones" formula from combinatorics tells us that $f(x_{0/i}, \ldots, x_{n/i})$ has $\binom{n+m}{n}$ coefficients. Hence, we conclude that

$$\dim \Gamma(\mathbb{P}^n, \mathscr{O}(m)) = \binom{n+m}{n}.$$

For $m < 0$ it is clear that f cannot be polynomial, so $\mathscr{O}(m)$ has no non-zero global sections and $\dim \Gamma(\mathbb{P}^n, \mathscr{O}(m)) = 0$.

Remark 2.1.2. Note that the condition

$$f\left(x_{0/i}, \ldots, \frac{1}{x_{i/j}}, \ldots, x_{n/i}\right) x_{i/j}^m = g(x_{0/j}, \ldots, x_{n/j})$$

is independent of scaling.

It can be shown that these are in fact the **only** line bundles on $\mathbb{P}^n$. Another important fact is that the *Picard group* of $\mathbb{P}^n$, (the group of invertible sheaves on $\mathbb{P}^n$ up to isomorphism) $\mathrm{Pic}(\mathbb{P}^n)$ is generated by $\mathscr{O}(1)$. Furthermore, $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

Theorem 2.1.3. *The automorphism group of* $\mathbb{P}^n$ *is* $\mathrm{PGL}_{n+1}(\mathbb{C})$

Proof. Suppose $\pi : \mathbb{P}^n \to \mathbb{P}^n$ is an automorphism. Note that $\pi^*(\mathscr{O}(1))$ must be a line bundle on $\mathbb{P}^n$, hence, π induces an automorphism of $\mathrm{Pic}(\mathbb{P}^n)$ with inverse $(\pi^{-1})^*$. Such an automorphism must send the generator of $\mathrm{Pic}(\mathbb{P}^n)$ to another generator. Since $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n)$ has generators 1 and -1, $\pi^*(\mathscr{O}(1))$ must be either $\mathscr{O}(1)$ or $\mathscr{O}(-1)$. We have shown above that $\mathscr{O}(-1)$ has no non-zero global sections, so we must have $\pi^*(\mathscr{O}(1)) = \mathscr{O}(1)$. Furthermore, out above calculations show that $\dim \Gamma(\mathbb{P}^n, \mathscr{O}(1)) = n + 1$ and that the global sections are determined up to scalars. Hence,

$$\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C}).$$

□

2.2 The Automorphism Group of an Arbitrary Variety

For a scheme X, a *family of automorphisms of* X *parameterized by a scheme* S is an automorphism of the S-scheme $X \times S$.

Remark 2.2.1. Suppose g is an automorphism of $X \times S$ then g satisfies the relation

$$g(x, s) = (f(x, s), x), \quad \text{for all } x \in X, s \in S$$

where $f : X \times S \to S$ is a morphism of schemes such that for each fixed $s \in S(\mathbb{C})$

$$\begin{aligned} f_s : X &\longrightarrow X \\ x &\longmapsto f(x, s) \end{aligned}$$

is an automorphism of X. The families of automorphisms of X paramaterised by S form a group $\mathrm{Aut}(X \times S/S)$.

We can move between families automorphisms of X paramaterised by a scheme S to families paramaterised by another scheme S'. Given $g = f \times \mathrm{id}_S \in \mathrm{Aut}(X \times S/S)$ and a morphism of schemes $u : S' \to S$ we have an induced map

$$\begin{aligned} g' : X \times S' &\longrightarrow S' \\ (x, s') &\longmapsto (f(x, u(s')), s') \end{aligned}$$

which is a family of automorphisms of X paramaterised by S', i.e. $g' \in \mathrm{Aut}(X \times S'/S')$. Furthermore, pulling back u gives us the group homomorphism

$$\begin{aligned} u^* : \mathrm{Aut}(X \times S/S) &\longrightarrow \mathrm{Aut}(X \times S'/S') \\ g &\longmapsto g'. \end{aligned}$$

Definition 2.2.2. Let X be a scheme whose family of automorphisms is paramaterised by S. The *automorphism group functor* is a contravariant functor from the category of schemes to the category of groups defined as

$$\begin{aligned} \mathbf{Aut}_X : \mathrm{Sch} &\longrightarrow \mathrm{Grp} \\ S &\longmapsto \mathrm{Aut}(X \times S/S) \\ u &\longmapsto u^*. \end{aligned}$$

In order to understand Matsumura-Oort, we first want to expand our notion of algebraic group. Firstly, we define an algebraic $\mathbb{C}$-scheme as a scheme of finite type over $\mathbb{C}$. Furthermore, we let $* = \mathrm{MaxSpec}(\mathbb{C})$, the set of maximal ideals of $\mathbb{C}$.

Definition 2.2.3. Let G be an algebraic $\mathbb{C}$-scheme and $m : G \times G \to G$ be a morphism of schemes. We call the pair (G, m) an *algebraic group scheme* (which we may refer to simply as an *algebraic group*) over $\mathbb{C}$ if there exist morphisms $e : * \to G$ and $\mathrm{inv} : G \to G$ such that the following diagrams commute:

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{\mathrm{id}\times m} & G\times G \\ {\scriptstyle m\times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ G\times G & \xrightarrow[m]{} & G \end{array} \qquad \begin{array}{ccccc} *\times G & \xrightarrow{e\times \mathrm{id}} & G\times G & \xleftarrow{\mathrm{id}\times e} & G\times * \\ & {\scriptstyle\cong}\searrow & \big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle\cong} & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv},\mathrm{id})} & G\times G & \xleftarrow{(\mathrm{id},\mathrm{inv})} & G \\ \big\downarrow & & \big\downarrow{\scriptstyle m} & & \big\downarrow \\ * & \xrightarrow{e} & G & \xleftarrow{e} & * \end{array}$$

A *locally algebraic group* is a group scheme, locally of finite type.

Theorem 2.2.4 (Matsumura-Oort, [MO67]). *If* X *is a proper scheme then the functor* $\mathbf{Aut}_X$ *is represented by a locally algebraic group.*

Corollary 2.2.5. *If* X *is a connected projective variety then* Aut(X) *is a connected algebraic group whose Lie algebra is* $\Gamma(X, \mathscr{T}_X)$.

Proof. Since X is projective, we have a closed immersion into $\mathbb{P}^n$. Hence, $X \to \mathbb{P}^n$ is proper. Then, by the Valuative Criterion of Properness (see [Har77]) $\mathbb{P}^n$ is proper over $\mathrm{Spec}(\mathbb{C})$. Hence, the composition $X \to \mathrm{Spec}(\mathbb{C})$ is proper. So X is a proper scheme and the Matsumura-Oort theorem tells us that $\mathbf{Aut}_X$ is represented by a locally algebraic group. Hence, by [Bri18, p. 5], the automorphism group (scheme) of X, Aut_X, is a locally finite type group scheme. By [Sta24, Proposition 0B7R] we see that Aut^0_X is quasi-compact and thus Aut^0_X is a finite type group scheme i.e. an algebraic group. As X is connected $\mathrm{Aut}^0_X = \mathrm{Aut}_X$ is an algebraic group. Furthermore, by [Bri18, p. 6], the Lie algebra of Aut_X consists of derivations of the structure sheaf $\mathscr{O}_X$ hence

$$\mathrm{Lie}(\mathrm{Aut}(X)) = \Gamma(X, \mathscr{T}_X).$$

□

3 The Borel-Remmert Theorem

In this chapter we will discuss a result by Borel and Remmert. The theorem holds for Kähler manifolds, however we need not define this as every smooth complex projective variety is a Kähler manifold. We will take X to be a homogeneous, connected, smooth projective variety.

The goal of this chapter is to decompose X into an abelian variety and a rational (birational to $\mathbb{P}^n$) homogeneous variety. We will call such a decomposition the *Borel-Remmert decomposition* of X and write

$$X \cong A \times G/P$$

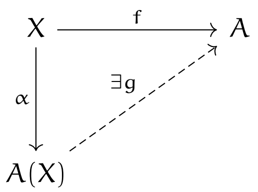
for the decomposition.

Theorem 3.0.1 (Borel-Remmert,[BR62]). *Let X be a connected, homogeneous and smooth projective variety. Then X is isomorphic to the product $A \times G/P$, where A is an abelian variety and P parabolic.*

3.1 Abelian Varieties

First, we need to set out a notion of the *abelianisation* of a variety. We do this by contructing *Albanese variety* associated to a variety X. Named after Giacomo Albanese, we construct the Albanese variety in the following way.

Definition 3.1.1. Let X be a variety with a basepoint. The *Albanese variety*, $A(X)$, is defined by the following universal property: There exists a morphism $\alpha : X \to A(X)$ (of pointed varieties) called the *Albanese map* such that for any map of pointed varieties $f : X \to A$ where A is abelian, there exists a $g : A(X) \to A$ such that the diagram



commutes.

We can also describe the Albanese explicitly as follows: Suppose $X \subseteq \mathbb{P}^n$ is a variety with basepoint p then

$$A(X) = H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z}).$$

Furthermore, if $q \in X$ and $\mu \in H^0(X, \Omega^1_X)$ we define the Albanese map $\alpha : X \to A(X)$ via

$$q \mapsto \left[x \mapsto \int_p^q x\right]$$

where $\int_p^q x$ is defined up to cycles $H_1(X, \mathbb{Z})$.

We will use without proof the following theorem by Chevalley, which gives abelian varieties the structure of an algebraic group. A modern proof can be found in [Con02].

Theorem 3.1.2 (Chevalley's Structure Theorem)**.** *Let* G *be an algebraic group, then there exists a unique closed connected normal affine subgroup* $G_{\text{aff}} \subseteq G$ *such that* G/G_{aff} *is an abelian variety. Moreover,* $A(G) = G/G_{\text{aff}}$.

3.2 Tangent Sheaves

There is a nice algebraic formulation of the tangent space at the identity of a connected algebraic group.

Definition 3.2.1. Let G be a connected algebraic group, the tangent space at e, $\mathfrak{g} := T_e G$, is the *Lie algebra* of G. Furthermore, if X is a G-space $\mathfrak{g}$ acts on X by vector fields. For smooth varieties, we denote the *tangent sheaf* $\mathscr{T}_X$ and have an induced morphism of Lie algebras

$$\begin{aligned} \mathrm{op}_X : \mathfrak{g} &\longrightarrow \Gamma(X, \mathscr{T}_X) \\ \xi &\longmapsto \nu_\xi \end{aligned}$$

where ν_ξ is the vector field induced by the action of G on X. Specifically, for each $x \in X$

$$\nu_\xi(x) = \left.\frac{d}{dt}\right|_{t=0} (\exp(-t\xi)x).$$

It follows that we have a corresponding map on sheaves

$$\underline{\mathrm{op}_X} : \mathscr{O}_X \otimes \mathfrak{g} \longrightarrow \mathscr{T}_X.$$

Consider the action $G \times X \to X$ where X is a homogeneous G-space. By assumption the group action is surjective so $g_1 = g_2$ if and only if $g_1 x = g_2 x$ which gives us $g_2^{-1} g_1 x = x$, i.e. $g_1 G_x = g_2 G_x$. Now consider a map

$$\begin{aligned} \varphi : G/G_x &\longrightarrow X \\ gG_x &\longmapsto gx. \end{aligned}$$

Note that φ is surjective by the transitivity of the action of G on X. Furthermore,

$$\begin{aligned} \varphi(g_1 G_x) &= g_1 x \\ &= g_2 x \\ &= \varphi(g_2 G_x) \\ \implies g_1 G &= g_2 G. \end{aligned}$$

So φ is injective and moreover is an isomorphism of G-spaces.

Remark 3.2.2. For a homogeneous G-space X, we have an isomorphism of pointed G-spaces $(G/G_x, eG_x) \cong (X, x)$.

Recall that, by Mastumura-Oort, the Lie algebra of the automorphism group of a projective variety is $\Gamma(X, \mathscr{T}_X)$. We then have the following useful lemma.

Lemma 3.2.3 ([Bri12]). *A connected projective variety X is homogeneous if and only if*

$$\underline{\mathrm{op}_X} : \mathscr{O}_X \otimes \Gamma(X, \mathscr{T}_X) \to \mathscr{T}_X$$

is surjective.

Proof. For the forwards direction, suppse X is homogeneous and choose arbitrarily a base-point $x \in X$. As X is homogeneous, the orbit map

$$\begin{aligned} \varphi_x : G &\longrightarrow X \\ g &\longmapsto g.x \end{aligned}$$

is surjective. Moreover, the differential of φ_x

$$d\varphi_x : \mathfrak{g} \longrightarrow T_x X$$

is also surjective. However, $d\varphi_x$ is a map on stalks of $\mathscr{T}_X$. More precisely

$$\varphi_x = \left(\underline{\mathrm{op}_X}\right)_x .$$

Since we picked x arbitrarily, it follows that $\underline{\mathrm{op}_X}$ is surjective.

For the converse, assume $\underline{\mathrm{op}_X}$ is surjective. Let $G = \mathrm{Aut}(X)$. Corollary 2.2.5 tells us that $\mathfrak{g} = \Gamma(X, \mathscr{T}_X)$. As $\underline{\mathrm{op}_X}$ is surjective, the induced map on stalks

$$\left(\underline{\mathrm{op}_X}\right)_x : \Gamma(X, \mathscr{T}_X) \to T_x X$$

is surjective too. Hence, $d\varphi_x$ is surjective for each $x \in X$, furthermore, φ_x is a *submersion* whose image $G \cdot x$ is open in X. Since x is arbitrary and X is connected we conclude X is homogeneous. □

Lemma 3.2.4 ([Bri12]). *If X is projective with trivial tangent bundle $\mathscr{T}_X$, then X is abelian.*

Proof. Our previous lemma shows X is homogeneous, hence X is of the form G/H where $G = \mathrm{Aut}(X)$ and H is the stabiliser of some point in X. Since $\mathscr{T}_X$ is trivial, we have that

$$\dim(X) = \dim(\Gamma(X, \mathscr{T}_X)) = \dim(\mathfrak{g}) = \dim(G)$$

so H is finite and X is abelian. Furthermore, H acts trivially on X so $H = \{e\}$. □

Lemma 3.2.5 ([Bri12]). *Any connected algebraic group can be written as $G = G_{\mathrm{aff}} Z(G)$*

3.3 A Proof of the Borel-Remmert Theorem

We are now ready to prove Theorem 3.0.1.

Proof of the Borel-Remmert Theorem. Let $G = \mathrm{Aut}(X)$. By Theorem 1.2.6 $Z(G)$ has a fixed point. Furthermore, $Z(G)$ is a normal subgroup of G. Our previous arguments show that, as X is homogeneous, $Z(G) = \{e\}$. Chevalley's structure theorem tells us that $A := Z(G)$ is abelian and $G_{\mathrm{aff}} \cap A$ is finite.

The previous lemma tells us that

$$\begin{aligned} G_{\mathrm{aff}} \times A &\longrightarrow G \\ (g, a) &\longmapsto ga^{-1} \end{aligned}$$

is a surjective morphism of algebraic groups whose kernel is $G_{\mathrm{aff}} \cap A$. Hence, $G \cong (G_{\mathrm{aff}} \times A)/K$ for some finite subgroup $K \subseteq Z(G)$.

Theorem 1.2.6 again tells us that $R(G_{\mathrm{aff}})$ has a fixed point, so it acts trivially. Thus, we may assume G_{aff} is semisimple. By similar arguments $Z(G_{\mathrm{aff}})$ is trivial and G_{aff} is adjoint. We conclude that $G_{\mathrm{aff}} \cap A$ is trivial, i.e. $K = \{e\}$. Hence, $G = G_{\mathrm{aff}} \times A$.

Matsumura-Oort tells us that for $x \in X$, G_x is affine and thus $G_x \subseteq G_{\mathrm{aff}}$. Since G/G_x is complete, so is G/G_x and G_{aff}/G_x. By Theorem 1.2.4 $P := G_x$ is parabolic in G_{aff}.

Finally, consider the projection map $G_{\mathrm{aff}} \times A \to G_{\mathrm{aff}}$ and its restriction $\pi : G_x \to G_{\mathrm{aff}}$ whose kernel is $A \cap G_x$. We note that $A \cap G_x$ is normal and this trivial. Since $[\pi(G_x) : P]$ is finite and P is parabolic we know that P is connected, equal to its normaliser and $\pi(G_x) = P$. Thus, $G_x = P$ and we have shown that

$$X = G_{\text{aff}}/P \times A.$$

□

4 The Van de Ven Theorem

Van de Ven stated the following theorem: The only compact submanifolds with splitting tangent sequence of the projective space are linear subspaces.

In 1958 Van de Ven published the proof of his theorem in [Ven58]. Since then many proofs have been found. In this chapter we will present a proof by Mustaţă and Popa ([MP96]), published in 1996.

Definition 4.0.1. A *short exact sequence* is a sequence

$$0 \to A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \to 0$$

such that $\operatorname{im} f = \ker g$.

Theorem 4.0.2 (The Van de Ven Theorem for $\mathbb{P}^n$)**.** *Let* X *be a smooth closed subvariety of* $\mathbb{P}^n$*. Let* T_X *and* $N_{X|\mathbb{P}^n}$ *be the tangent and normal bundle of* X*. Then the short exact sequence*

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X|\mathbb{P}^n} \to 0$$

splits if and only if X *is a linear subvariety of* $\mathbb{P}^n$*. Written explicitly, the above sequence is isomorphic to the sequence*

$$0 \to T_X \to T_X \oplus N_{X|\mathbb{P}^n} \to N_{X|\mathbb{P}^n} \to 0$$

that is, there exist morphisms f, i, π *such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbb{P}^n}|_X & \longrightarrow & N_{X|\mathbb{P}^n} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & T_X & \xrightarrow{i} & T_X \oplus N_{X|\mathbb{P}^n} & \xrightarrow{\pi} & N_{X|\mathbb{P}^n} & \longrightarrow & 0 \end{array}$$

commutes.

4.1 Preliminaries

Before we prove Theorem 4.0.2, we will first define some scheme-theoretic tools. These tools are standard and are available in more detail in [Har77]. Our goal is to restate (1) in terms of schemes and use their cohomological properties.

Definition 4.1.1. Let $f : (X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces. Suppose that $f : X \to Y$ is a continuous map, define the *direct image sheaf* $f_*\mathscr{O}_X$ on Y by $f_*\mathscr{O}_X(V) = \mathscr{O}_X\left(f^{-1}(V)\right)$ for all $V \subseteq Y$ open.

Definition 4.1.2. Let X be a topological space and $f : \mathscr{O}_1 \to \mathscr{O}_2$ be a morphism of sheaves of rings on X. Let $\mathscr{F}$ be an $\mathscr{O}_2$ module. An f-*derivation into* $\mathscr{F}$ is a map

$$D : \mathscr{O}_2 \to \mathscr{F}$$

that annihilates $\mathrm{im}(f)$, is additive and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b.$$

We now introduce one of our main tools for the proof of Theorem 4.0.2, a global analogue of an ideal of a ring. We are then able to express tangent bundles in terms of sheaves.

Definition 4.1.3. Let X be a closed subscheme of Y, and let $i : X \hookrightarrow Y$ be the inclusion morphism. We have an induced map of sheaves

$$i^\# : \mathscr{O}_Y \to i_*\mathscr{O}_X$$

whose kernel we call the *ideal sheaf*, $\mathscr{I}_X$ of X. When the context is clear we will simply write $\mathscr{I}$ for the sheaf of ideals. We call $\mathscr{I}/\mathscr{I}^2$ the *conormal sheaf* of X in Y and its dual $\mathscr{N}_{X|Y} = \mathrm{hom}_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X)$ the *normal sheaf* of X in Y. Furthermore, we define the *first infinitesimal neighbourhood* of X in Y to be $X^{(1)} := (X, i^\#(\mathscr{O}_Y)/\mathscr{I}^2)$.

Note that X is a closed subscheme of $X^{(1)}$, so there is a cannonical inclusion map $X \hookrightarrow X^{(1)}$ corresponding to the projection map of sheaves.

Now that we have the tools we need, we are ready to approach Van de Ven. We first rephrase the statement to be about the splitting in the language of sheaves, that is if the sequence

$$0 \to \mathscr{T}_X \to \mathscr{T}_{\mathbb{P}^n}|_X \to \mathscr{N}_{X|\mathbb{P}^n} \to 0 \tag{1}$$

splits then X is a linear subvariety of $\mathbb{P}^n$. Dualising gives us the sequence

$$0 \to \mathscr{N}^\vee_{X|\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n|X} \to \Omega^1_X \to 0, \tag{2}$$

where Ω^1_X is the *cotangent sheaf* on X. From now on we will use the terms (co)tangent bundle and (co)tangent sheaf interchangably.

We will take without proof that (2) is short exact (thus so is (1)). A proof can be found in [Vak24, Section 21.2.]

A well known and important theorem is the exactness of the *Euler sequence*. We will use the result without proof, as it is a standard result. Proofs of different flavours can be found in [Har77] and [Vak24].

Theorem 4.1.4 (The Euler Exact Sequenece)**.** *The cotangent bundle $\Omega^1_{\mathbb{P}^n}$ satisfies the following exact sequence*

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow 0.$$

The Euler sequence gives us a more concrete grasp on the cotangent bundle of $\mathbb{P}^n$. We will be use it directly to prove Van de Ven's theorem for the case $\dim(X) = 1$.

The final piece of setup we need are some properties relating to the (algebraic) group of invertible sheaves, the Picard group $\text{Pic}(X) \cong H^1(X, \mathscr{O}_X^*)$. Firstly, we introduce the Néron-Severi group.

Definition 4.1.5. Let $\text{Pic}^0(X)$ be the subgroup of $\text{Pic}(X)$ consisting of divisors algebraically equivalent to 0. We define that *Néron-Severi group* as $\text{NS}(X) := \text{Pic}(X)/\text{Pic}^0(X)$.

Theorem 4.1.6. *There exists an injective map*

$$\text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^1(X, \Omega^1_X).$$

Proof. The exponential sequnce

$$0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^* \to 1$$

induces a long exact piece

$$\cdots \to H^1(X, \mathscr{O}_X) \to \text{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \cdots$$

Here c_1 is the *first Chern class map* which identifies a line bundle with its *first chern class.* It is well known that $\text{im}(c_1)$ is the Néron-Severi group since $\ker(c_1) = \text{Pic}^0(X)$ where $\text{Pic}^0(X)$ is the subgroup of divisors algebraically equivalent to zero. Furthermore,

$$\text{im}(c_1) = H^{(1,1)}(X) \cap H^2(X, \mathbb{Z})$$

so $\text{NS}(X)$ is a discrete subgroup of $H^{(1,1)}(X)$. Finally, by Dolbeault's theorem

$$H^{(1,1)}(X) \cong H^1(X, \Omega^1_X)$$

so by tensoring with $\mathbb{C}$ we obtain an injective map

$$\text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C} \hookrightarrow H^1(X, \Omega^1_X).$$

□

4.2 Setup

Our first lemma is this section is the bread and butter of our proof. Our arguments on the first infinitesimal neighbourhood show us that, in order for the cotangent sequence to split, the first order approximations of $\mathbb{P}^n$ and X must coincide for some neighbourhood of X.

Lemma 4.2.1. *The morphism* δ *in the exact sequence of sheaves*

$$0 \to \mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega^1_{Y|X} \to \Omega^1_X \to 0$$

admits a left inverse if and only if the inclusion $i : X \hookrightarrow X^{(1)}$ *admits a retraction (a left inverse).*

Proof. We begin by simplifying each direction in the proof. Let $\pi : \mathscr{O}_Y/\mathscr{I}^2 \to \mathscr{O}_Y/\mathscr{I}$ denote the cannonical projection map on quotients. The existence of a retract of i is equivalent to the existence of a ring homomorphism θ such that $\pi \circ \theta = \mathrm{id}_{\mathscr{O}_Y/\mathscr{I}}$.

However, the existence of a left inverse of δ is equivalent to the existence of a derivation $D : \mathscr{O}_Y \to \mathscr{I}/\mathscr{I}^2$ such that $D|_{\mathscr{I}} = p$ where p is the projection map $\mathscr{I} \to \mathscr{I}/\mathscr{I}^2$.

We are now ready to prove the lemma. Let $q : \mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{I}^2$ be the cannonical projection. Suppose we have a derivation D such that $D|_{\mathscr{I}} = p$. Define $\theta : \mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{I}^2$ as $\theta = q - D$. We evaluate θ at $\mathscr{I}$ to obtain

$$\theta(\mathscr{I}) = q(\mathscr{I}) - D(\mathscr{I}) = q(\mathscr{I}) - p(\mathscr{I}) = \mathscr{I}/\mathscr{I}^2 - \mathscr{I}/\mathscr{I}^2 = 0_{\mathscr{O}_Y/\mathscr{I}^2}.$$

Thus, θ factors through $\mathscr{O}_Y/\mathscr{I} \to \mathscr{O}_Y/\mathscr{I}^2$, call this map $\overline{\theta}$. We then see that $\pi \circ \overline{\theta} = \mathrm{id}_{\mathscr{O}_Y/\mathscr{I}}$. For the other direction, given $\overline{\theta}$ we obtain $D = \overline{\theta} - q$. □

We can now prove one direction (the much easier direction) of Van de Ven's theorem. The proof is rather short since $\mathbb{P}^n$ has a linear structure and so will any $\mathbb{P}^r$ embedded linearly.

Theorem 4.2.2. *Suppose* $X \subseteq \mathbb{P}^n$ *be an* r*-dimensional linear subspace. Then, the sequence*

$$0 \to \mathscr{N}^\vee_{X|\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n|X} \to \Omega^1_X \to 0$$

splits.

Proof. Let $X' \subseteq \mathbb{P}^n$ be an $(n - r - 1)$-dimensional subspace such that $X \cap X' = \varnothing$ and let $U = \mathbb{P}^n \setminus X'$ and π denote the projection $U \to X$. It is clear that

$$X \subseteq X^{(1)} \subseteq U$$

and that $\pi|_{X^{(1)}}$ is the retract of $i : X \hookrightarrow X^{(1)}$. By Lemma 4.2.1 the sequence splits. □

Lemma 4.2.3. *Let* L *be a linear subspace of* $\mathbb{P}^n$ *such that* $X \subseteq L \subseteq \mathbb{P}^n$. *We have the following commutative diagram of vector bundles.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{N}^{\vee}_{X|\mathbb{P}^n} & \longrightarrow & \Omega^1_{\mathbb{P}^n}|_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{N}^{\vee}_{X|L} & \longrightarrow & \Omega^1_L|_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0
\end{array}$$

Suppose that the top sequence splits, then so does the bottom sequence.

Proof. Firstly, note that the vertical maps are restriction maps induced by the inclusion $L \subseteq \mathbb{P}^n$. If the top sequence splits, we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{N}^{\vee}_{X|\mathbb{P}^n} & \hookrightarrow & \mathscr{N}^{\vee}_{X|\mathbb{P}^n} \oplus \Omega^1_X & \twoheadrightarrow & \Omega^1_X & \longrightarrow & 0 \\
 & & \| & & \cong\downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{N}^{\vee}_{X|\mathbb{P}^n} & \longrightarrow & \Omega^1_{\mathbb{P}^n}|_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{N}^{\vee}_{X|L} & \longrightarrow & \Omega^1_L|_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\
 & & \| & & \exists\cong \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{N}^{\vee}_{X|L} & \hookrightarrow & \mathscr{N}^{\vee}_{X|L} \oplus \Omega^1_X & \twoheadrightarrow & \Omega^1_X & \longrightarrow & 0
\end{array}$$

The isomorphism we want to show exists must make the diagram commute. Of course, we can see such an isomorphism exists by applying the restriction map $\mathscr{N}^{\vee}_{X|\mathbb{P}^n} \oplus \Omega^1_X \to \mathscr{N}^{\vee}_{X|L} \oplus \Omega^1_X$. □

Lemma 4.2.4. *If (2) splits,* $X \subseteq \mathbb{P}^n$ *and* $\dim(X) \geqslant 2$ *then*

$$H^1(X, \mathscr{I}/\mathscr{I}^2) = 0.$$

Proof. As (2) splits, we have $H^1(X, \Omega^1_{\mathbb{P}^n}|_X) = H^1(X, \mathscr{I}/\mathscr{I}^2) \oplus H^1(X, \Omega^1_X)$, so it is enough to show that $h^1(X, \Omega^1_{\mathbb{P}^n}|_X) \leqslant h^1(X, \Omega^1_X)$. By restricting the twisted Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathscr{O}_{\mathbb{P}^n} \to 0$$

to X, we obtain the *restricted twisted Euler sequence*

$$0 \to \Omega_{\mathbb{P}^n}|_X \to \mathscr{O}_X(-1)^{\oplus(n+1)} \to \mathscr{O}_X \to 0.$$

Taking cohomology gives us the long exact sequence

$$0 \longrightarrow H^0(X, \Omega^1_{\mathbb{P}^n}|_X) \longrightarrow H^0(X, \mathscr{O}_X(-1)^{\oplus(n+1)}) \longrightarrow H^0(X, \mathscr{O}_X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1(X, \Omega^1_{\mathbb{P}^n}|_X) \longrightarrow H^1(X, \mathscr{O}_X(-1)^{\oplus(n+1)}) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow \cdots$$

Of course $H^0(X, \mathscr{O}_X) = \mathbb{C}$. Furthermore, as $\mathscr{O}_X(-1) \cong \mathscr{O}_X(1)^*$, we have

$$H^1(X, \mathscr{O}_X(-1)^{\oplus(n+1)}) \cong \left(H^1(X, \mathscr{O}_X(1)^*)\right)^{n+1}$$

which by Kodaira's vanishing theorem with $\dim(X) = 2$ and $\mathscr{L}^{-1} = \mathscr{O}_X(1)^*$ is zero. Hence, $h^1(X, \Omega^1_{\mathbb{P}^n}|_X) = 1$. By Theorem 4.1.6, we have an injective map

$$\alpha : \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C} \hookrightarrow H^1(X, \Omega^1_X)$$

thus $h^1(X, \Omega^1_X) \geqslant 1$.

□

Example 4.2.5. Embed $\mathbb{P}^1$ into $\mathbb{P}^2$ via $\mathscr{O}_{\mathbb{P}^1}(2)$. Written precisely, we have a map

$$\begin{aligned} \nu_2 : \mathbb{P}^1_{[x:y]} &\longrightarrow \mathbb{P}(\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2)) = \mathbb{P}^2_{[z_0:z_1:z_2]} \\ [x:y] &\longmapsto [x^2 : xy : y^2] \end{aligned}$$

namely the *Veronese embedding*. Note that ν_2 is a degree 2 map which is an isomorphism onto its image. Furthermore, $\mathrm{im}(\nu_2) = \mathbb{V}(z_0 z_2 - z_1^2)$ which is a plane conic. The cotangent sequence is thus

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(2) \to 0.$$

Assume for contradiction that the cotangent sequence splits. The Euler sequence for $\mathbb{P}^2$ is

$$0 \to \Omega^1_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to \mathscr{O}_{\mathbb{P}^2} \to 0.$$

Restricting to $\mathbb{P}^1$ gives us

$$0 \to \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \to \mathscr{O}_{\mathbb{P}^1} \to 0.$$

Now we twist the above sequence by 2 and obtain

$$0 \to \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1}(2) \to \mathscr{O}_{\mathbb{P}^1}^{\oplus 3} \to \mathscr{O}_{\mathbb{P}^1}(2) \to 0.$$

We have an induced long exact sequence on cohomology

$$0 \to H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1}(2)) \to H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}^{\oplus 3}) \to H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2)) \to \cdots$$

Since $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}^{\oplus 3}) \to H^0(\mathbb{P}^1, \mathscr{O}^1_{\mathbb{P}}(2))$ is an isomorphism we have

$$H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1}(2)) = 0$$

which contradicts the existence of a right inverse for the map $\Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(2)$. Hence, the (co)tangent sequence does not split.

4.3 Proof of the Van de Ven problem for $\mathbb{P}^n$

Proof of the Van de Ven Problem on $\mathbb{P}^n$ *for* $\dim(X) \geqslant 2$. By Theorem 4.2.2 we can take X to be nondegenerate, that is if X embedds into $\mathbb{P}^n$, X is not contained in any hyperplane. Our aim is to show that $X = \mathbb{P}^n$.

Firstly, we show that $H^0(X, \mathscr{I}/\mathscr{I}^2) = 0$. We obtain the restricted twisted Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^n}|_X(1) \to \mathscr{O}_X^{\oplus(n+1)} \to \mathscr{O}_X(1) \to 0$$

which induces the cohomological chain

$$0 \to H^0(X, \Omega^1_{\mathbb{P}^n}|_X(1)) \to H^0(X, \mathscr{O}_X^{\oplus(n+1)}) \to H^0(X, \mathscr{O}_X(1)) \to \cdots$$

Note that $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}^{n+1}) \cong \mathbb{C}^n \cong H^0(X, \mathscr{O}_X^{n+1})$. Furthermore, the map

$$H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \longrightarrow H^0(X, \mathscr{O}_X(1))$$

is injective by the nondegeneration of X. Thus, $H^0\left(X, \Omega^1_{\mathbb{P}^n}|_X(1)\right) = 0$, which by the splitting of (2) yields $H^0(X, \mathscr{I}/\mathscr{I}^2(1)) = 0$.

By Lemma 4.2.1 there exists a morphism $r : X^{(1)} \to X$ such that $r \circ i = \mathrm{id}$. Let $L := r^*\mathscr{O}_X(1)$ be the pullback of the retract of $\mathscr{O}_X(1)$. By Lemma 4.2.4 $H^1(X, \mathscr{I}/\mathscr{I}^2) = 0$ and thus the map $i^* : H^1\left(X^{(1)}, \mathscr{O}_{X^{(1)}}\right) \to H^1(X, \mathscr{O}_X^*)$ is injective. Recall that Pic(X) is the group of isomorphism classes of line bundles, and we obtain the equivalent injective map

$$i^* : \mathrm{Pic}(X^{(1)}) \to \mathrm{Pic}(X).$$

As $r \circ i = \mathrm{id}$ we have $i^* \circ r^* = \mathrm{id}$ so

$$i^*L = i^*r^*\mathscr{O}_X(1) = \mathscr{O}_X(1).$$

Additionally, pulling back $\mathscr{O}_{X^{(1)}}(1)$ along the cannonical inclusion gives $i^*\mathscr{O}_{X^{(1)}}(1) \cong \mathscr{O}_X(1)$, and we obtain the isomorphism

$$L \cong \mathscr{O}_{X^{(1)}}(1).$$

Since each side of the above isomorphism differ only by multiplication by a nonzero constant we can take $L = \mathscr{O}_{X^{(1)}}(1)$.

We showed above that $H^0(X, \mathscr{I}/\mathscr{I}^2) = 0$, so the map

$$i^* : H^0(X^{(1)}, \mathscr{O}_{X^{(1)}}(1)) \to H^0(X, \mathscr{O}_X(1))$$

is injective. Furthermore, since $i^* \circ r^* = \mathrm{id}$, the composition

$$H^0(X, \mathscr{O}_X(1)) \xrightarrow{r^*} H^0(X^{(1)}, \mathscr{O}_{X^{(1)}}(1)) \xrightarrow{i^*} H^0(X, \mathscr{O}_X(1))$$

is the identity so i^* and r^* are isomorphisms. We hence have the following commutative diagram

$$\begin{array}{ccc} X^{(1)} & \hookrightarrow & \mathbb{P}(H^0(X^{(1)}, \mathscr{O}_{X^{(1)}}(1))^*) \\ {\scriptstyle r}\downarrow & & \cong\downarrow{\scriptstyle s} \\ X & \hookrightarrow & \mathbb{P}(H^0(X, \mathscr{O}_X(1))^*) \end{array}$$

where s is the isomorphism induced by r^*. Hence, r is an immersion and $\mathscr{I} = 0$. We conclude that $X = \mathbb{P}^n$. □

It remains only to prove the Van de Ven problem in the case $\dim(X) = 1$. We will follow the ideas of Laksov in [Lak81].

Proof of the case $\dim(X) = 1$. Let X be a non-degenerate curve in $\mathbb{P}^n$. Since $T_{\mathbb{P}^n}$ is ample, [Har66, Proposition (4.1)] gives us the restricted bundle $T_{\mathbb{P}^n}|_X$ is ample too. Assume for contradiction (1) splits $T_{\mathbb{P}^n}|_X \cong T_X \oplus N_{X|\mathbb{P}^n}$, by [Har66, Proposition (2.2)] each direct summand of $T_{\mathbb{P}^n}|_X$ we get T_X is an ample line bundle.

Laksov [Lak81] shows that this implies X is rational, hence we can take $X = \mathbb{P}^1 \subseteq \mathbb{P}^n$ as a degree $d \geqslant 2$ embedding. We then have the following standard commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{I}/\mathscr{I}^2 & \longrightarrow & \Omega^1_{\mathbb{P}^n}|_{\mathbb{P}^1} & \longrightarrow & \Omega^1_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^1}(-2) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{I}/\mathscr{I}^2 & \longrightarrow & \mathscr{O}_{\mathbb{P}^1}(-d)^{\oplus(n+1)} & \longrightarrow & E & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathscr{O}_{\mathbb{P}^1} & = & \mathscr{O}_{\mathbb{P}^1} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

which defines a rank two vector bundle $E \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$, $a_1, a_2 \in \mathbb{Z}$ on X. As the top row splits we have a map

$$(\mathcal{I}/\mathcal{I}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{\oplus(n+1)}$$

in particular an inclusion

$$\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-d)^{\oplus(n+1)}.$$

Twisting the inclusion by 2, we obtain

$$\mathcal{O}_{\mathbb{P}^1} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(-d+2)^{\oplus(n+1)}.$$

As $\mathcal{O}_{\mathbb{P}^1}$ has global sections, $\mathcal{O}_{\mathbb{P}^n}(-d+2)^{\oplus(n+1)}$ must do too. Hence, $-d+2 \geqslant 0$ so $d \leqslant 2$. However, since we assumed $d \geqslant 2$, we obtain $d = 2$. Thus, X is a rational curve of degree 2, which must be a plane conic. From here the proof is the same as in Example 4.2.5: Twisting the restricted Euler sequence by 2 gives us

$$0 \to \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1}(2) \to \mathcal{O}^3_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$

We then have the long exact sequence on cohomology:

$$0 \to H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}^3_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \to \cdots$$

Of course,

$$H^0(\mathbb{P}^1, \mathcal{O}^3_{\mathbb{P}^1}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$$

so $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1}(2)) = 0$, which contradicts the existence of a right inverse for the morphism $\Omega^1_{\mathbb{P}^2}|_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-2)$. □

5 Concluding Remarks

The Van de Ven problem for $\mathbb{P}^n$ is well documented and has many proofs. Likewise, the Borel-Remmert theorem has a few proofs e.g. the one covered in this dissertation and one in [Akh95]. However, the Matsumura-Oort theorem, from what the author has gathered, has seldom one proof with the exception of a sketch of the theorem for the case when X is projective by Brion in [Bri18].

Others have worked on the Van de Ven problem for other varieties. Namely, the Grassmannian variety and quadric hypersurfaces. We did do not have enough time to talk about those here but Jahnke has an excellent exposition in [Jah05].

On another note, the Van de Ven problem for the fundamental representations of the exceptional Lie group G_2 is only partially solved. One representation is realised as a quadric hypersurface and thus is covered in [Jah05]. The other fundamental representation, the adjoint one, is a more complicated space. It is the intersection of a few quadrics, for details see [Pro07]. The Van de Ven problem is currently unknown for the adjoint representation of G_2.

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